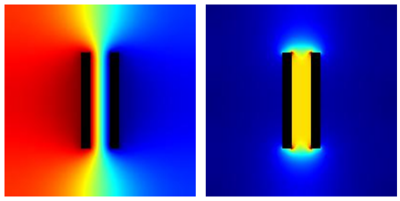
Liebmann technical documentation



Laplace equation 2D (XY)
(Cartesian coordinates)
relaxation scheme explained
(5 - point star)

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1 Liebmann technical documentation series

1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relaksacyjną Liebmanna. (Polish version / wersja polska)
2. Determination of electrostatic field distribution by using Liebmann relaxation method. (English version / wersja angielska)
3. Graphics. Mapping voltages to colours (colormaps).
4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme explained. (5 - point star)
5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme explained. (5 - point star)
6. Liebmann source sode. (ANSI C programming language)

2 Versions of this document

1. version 1 - 2023.11.03
2. version 2 - 2024.01.26
3. version 3 - 2024.02.02
4. version 4 - 2024.02.05
5. version 5 - 2024.05.18
6. version 6 - 2024.05.23
7. version 7 - 2024.05.24

3 Solving Laplace equation using relaxation method

I tried to solve Laplace equation using mainly information from Pierre Grivet's book (Electron Optics) - [1].
There are few editions of this book (1965, 1972). Second edition (1972) contains explanation of relaxation method (page 38).

More generalized approaches has been drafted by James R. Nagel - [2]. `https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/` (visited 2023-03-01).

There are also publications edited by Albert Septier: Focusing of Charged Particles [3] and Applied Charged Particle Optics (part A). [4].

I have also found some ideas in publication of D W O Heddle: Electrostatic Lens Systems [5] (especially using PC computers to solve electrostatic problems).

I have also found (brief) description of by - hand solving of Laplace equation by Bohdan Paszkowski - [6] (Polish edition). English translation of this book also exists - [7].

I would like to thank many people, who helped me with this challenge. Especially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis), who enabled me to use SIMION and MATLAB software while writing master's thesis about electron optical systems at University of Maria Curie - Skłodowska in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discussion about numerical methods. What is more, my colleague Bartosz in 2012 had explained me general problems with software efficiency. So he had also contributed significantly to the idea of Liebmann software (especially using C language).

4 Explanation of symbols in calculations

- P_i - i-th mesh node
- V_i - value of electrostatic potential at node P_i. Unit - $[\mathrm{V}]$
- h - mesh step (for example h_x - mesh step in x direction). Unit - $[\mathrm{mm}]$
- $g_{i+/-}$ - gradient in direction i (for example $g_{1x-} = \frac{V_1 - V_{1x-}}{h_x}$. Unit - $\left[\frac{\mathrm{V}}{\mathrm{mm}}\right]$
- i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, .., \mathrm{size\_row}$
- i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, .., \mathrm{size\_col}$

5 Mesh XY - type A

$h_x \neq h_y$
gradient V outside a mesh exists

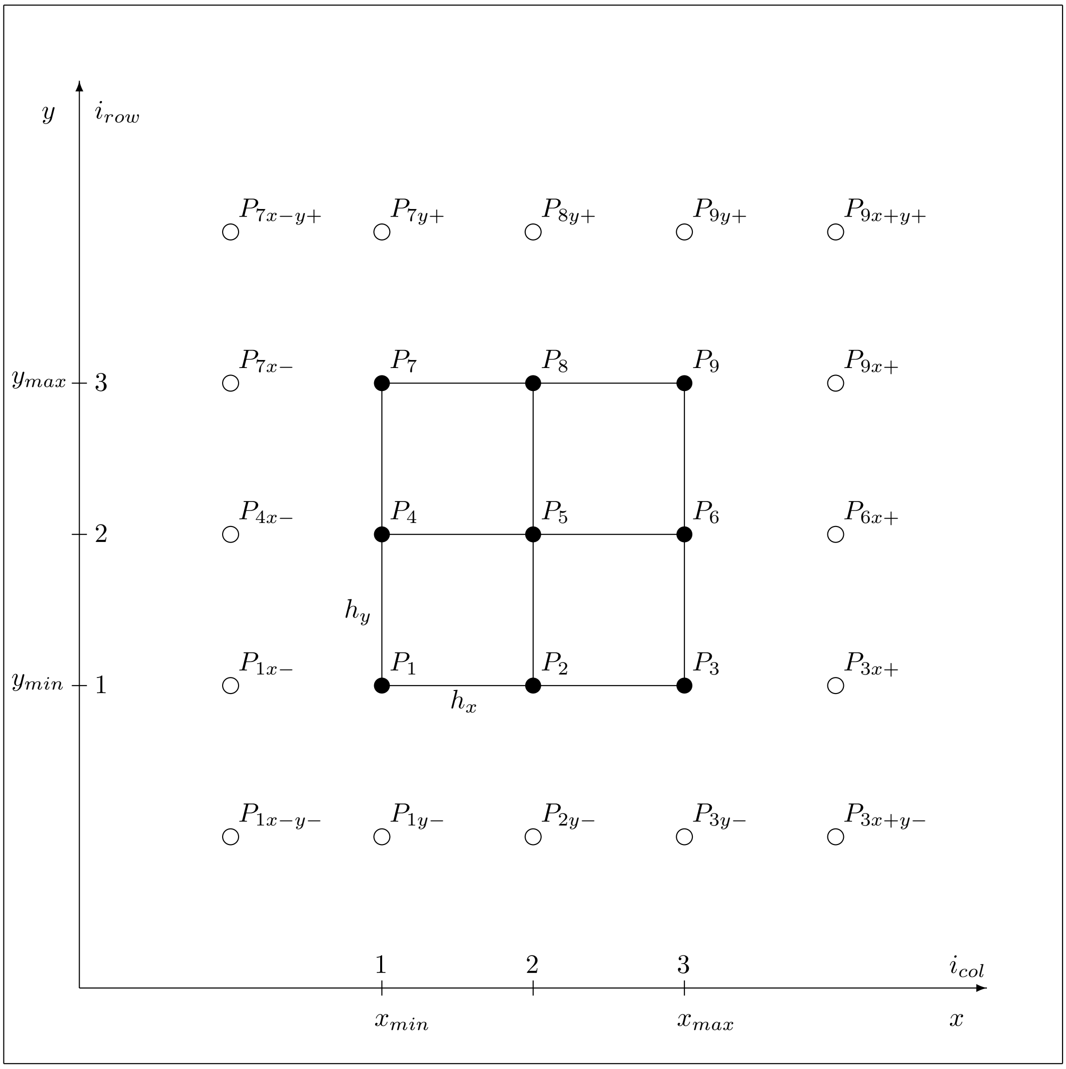


Figure 1: Mesh XY type A

6 Mesh XY - type B

$h_x \neq h_y$
gradient V outside a mesh does not exist

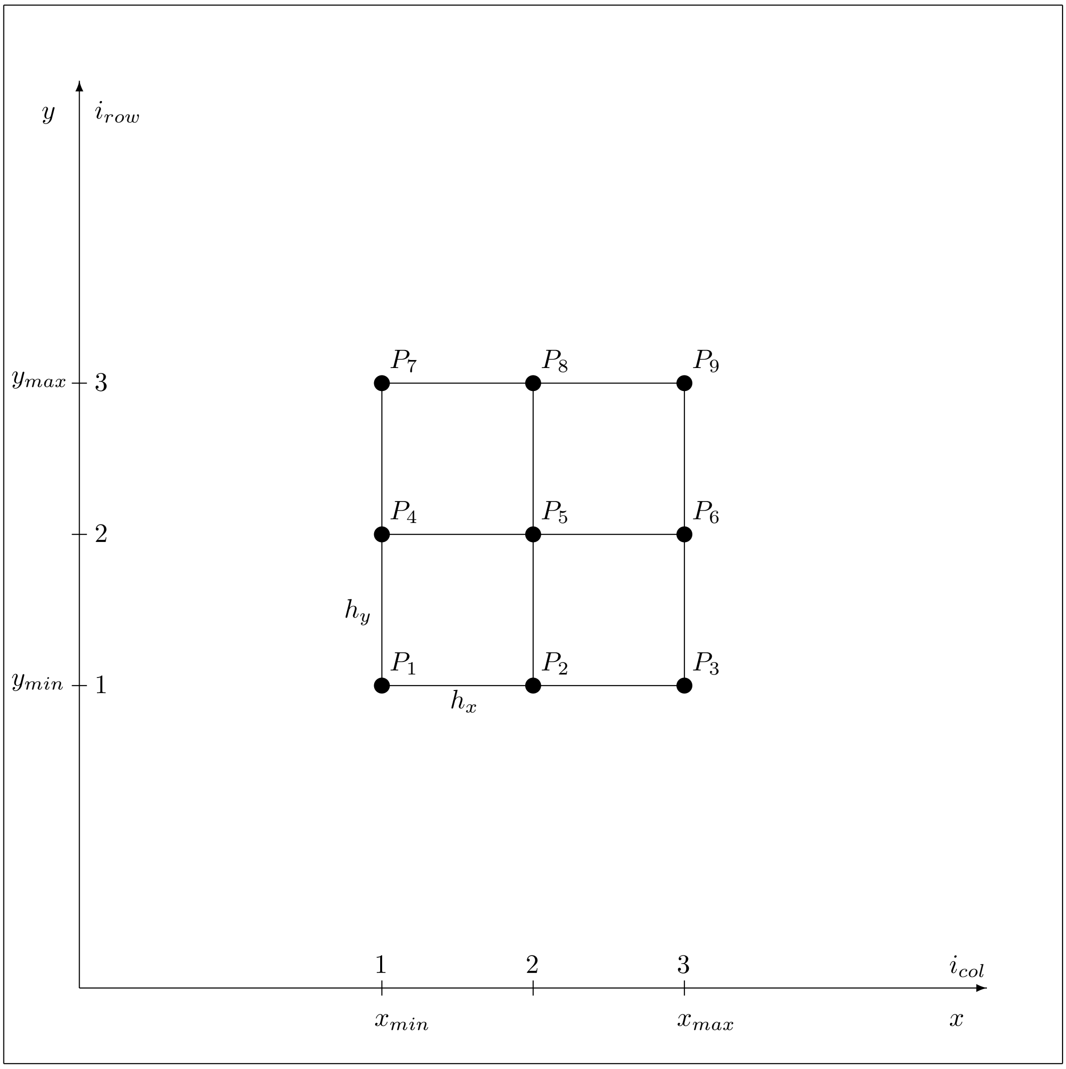


Figure 2: Mesh XY type B

7 Mesh XY - type C

$h_x = h_y = h$
gradient V outside a mesh exists

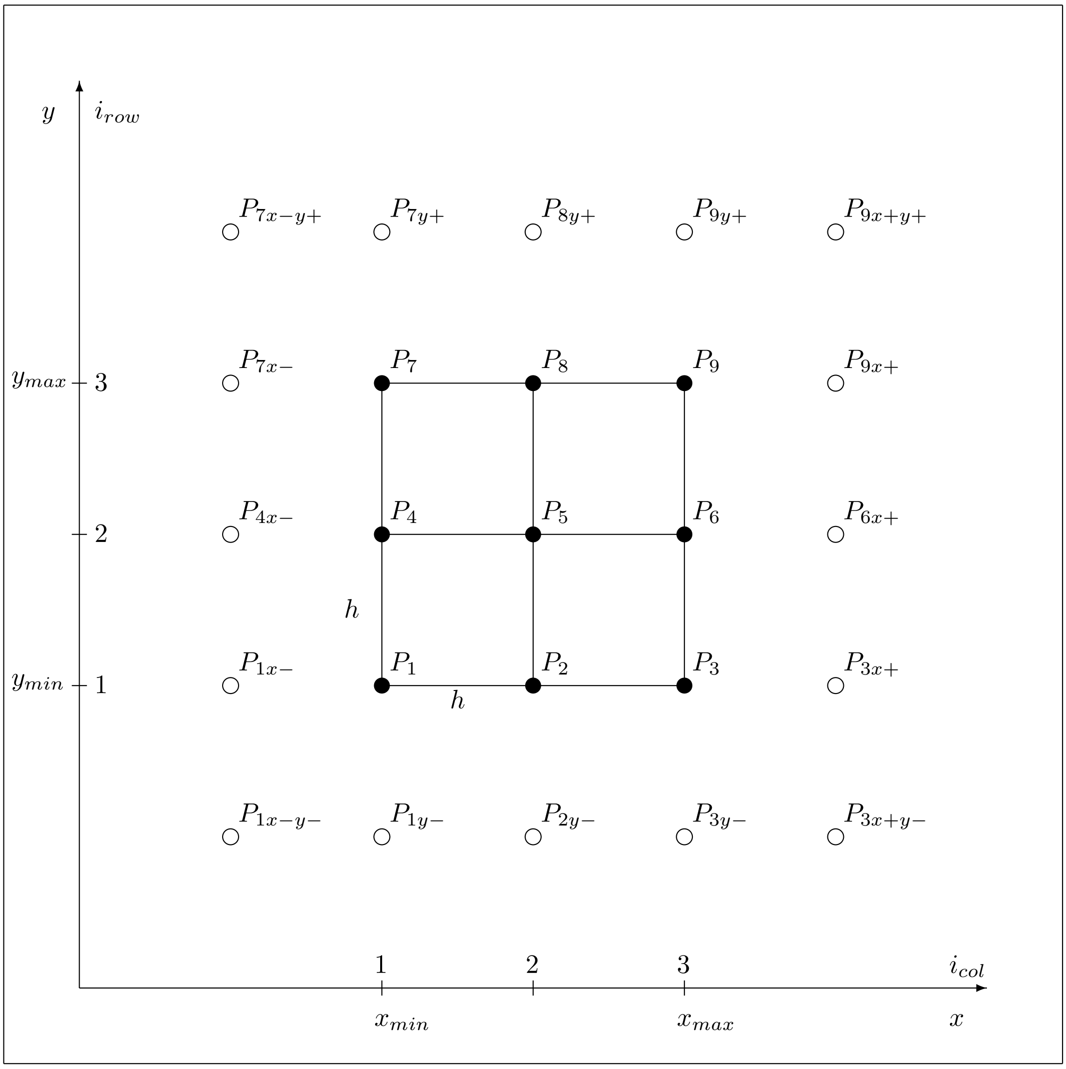


Figure 3: Mesh XY type C

8 Mesh XY - type D

$h_x = h_y = h$
gradient V outside a mesh does not exist

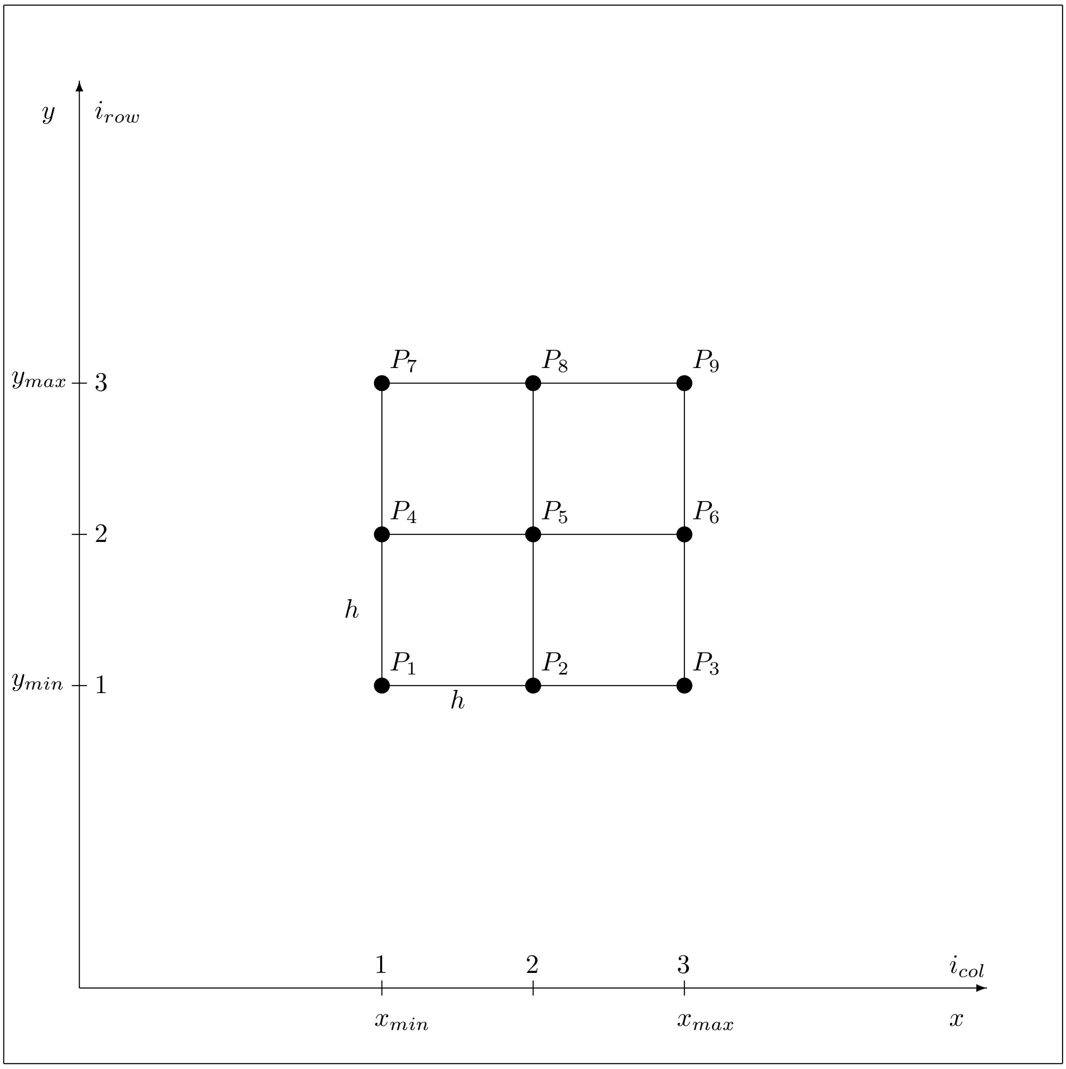


Figure 4: Mesh XY type D

9 Example of A-type mesh in ANSI C

Example of A- type mesh in ANSI C program. The mesh is represented by 2 dimensional array of double precision numbers. Rows and columns in mesh are numbered from 1 (this was my choice) instead of default 0 (as usual in C language). This choice nas pros and cons. Is is easier to calculate mesh size (size_row * size_col). Access to each node can be also more intuitive, but logic in each library function must contain this shift between node ordering styles.

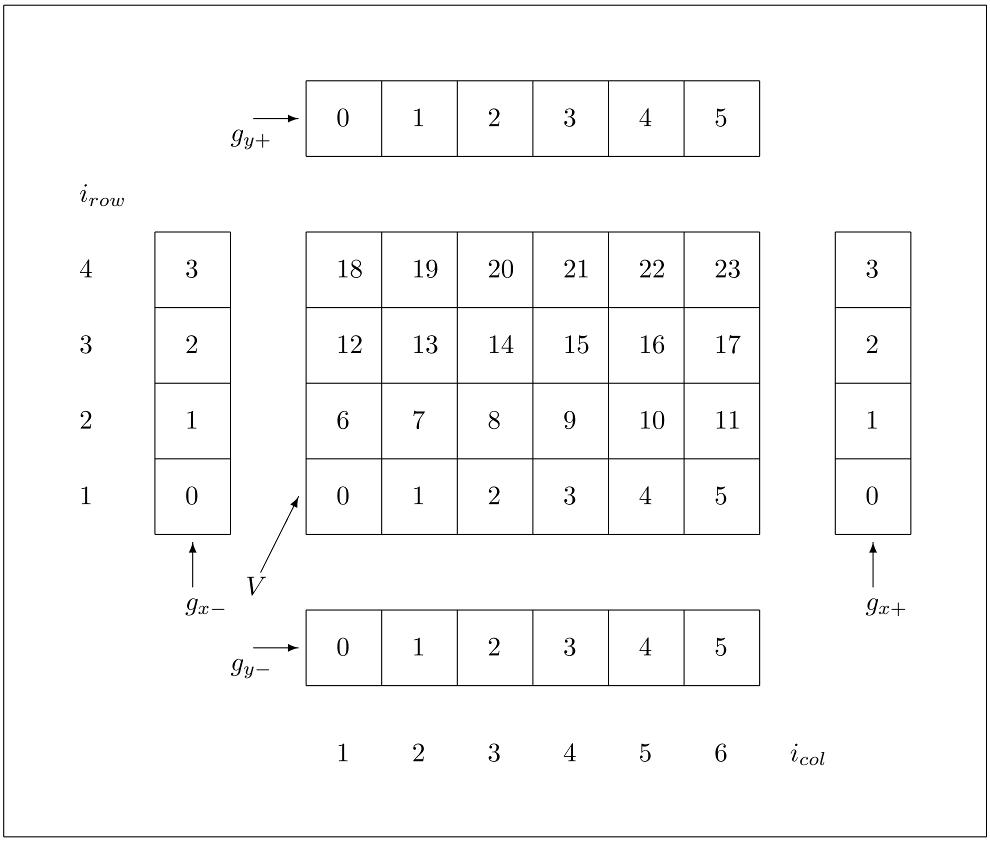
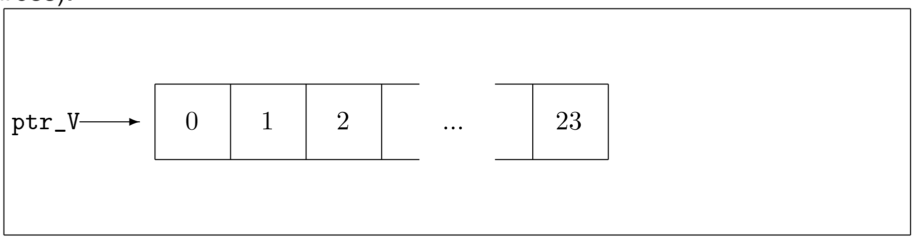


Figure 5: ANSI C - mesh XY type A

- $g_{x-} \equiv$ `double* ptr_gX_minus`
- $g_{x+} \equiv$ `double* ptr_gX_plus`
- $g_{y-} \equiv$ `double* ptr_gY_minus`
- $g_{y+} \equiv$ `double* ptr_gY_plus`
- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`

- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h_x == 1.0 [mm]`
- `double h_y == 2.0 [mm]`

The following picture describes analogous version of `ptr_V` mesh, which can be dynamically allocated on heap by pointer metod. The mesh is represented by single block of memory. The numbers or rows and columns are also known, so each node can be also accessed by appropriate index (memory address).



Each mesh point has its unique index (let's say `icp` - (index of central point)), which can be determined, if we know indices of row and column (`i_row`, `i_col`).

$$\texttt{icp == (i\_row - 1) * size\_col + i\_col - 1} \tag{9.1}$$

For example for each point of a mesh indices of row and column have values:

$$\begin{aligned}&\texttt{i\_row == 1, 2, ..\ \ , size\_row}\\&\texttt{i\_col == 1, 2, ..\ \ , size\_col}\end{aligned} \tag{9.2}$$

10 Example of B-type mesh in ANSI C

Example of B- type mesh in ANSI C program. The mesh is analogous to A - type mesh. There are no electric field gradients on mesh borders.

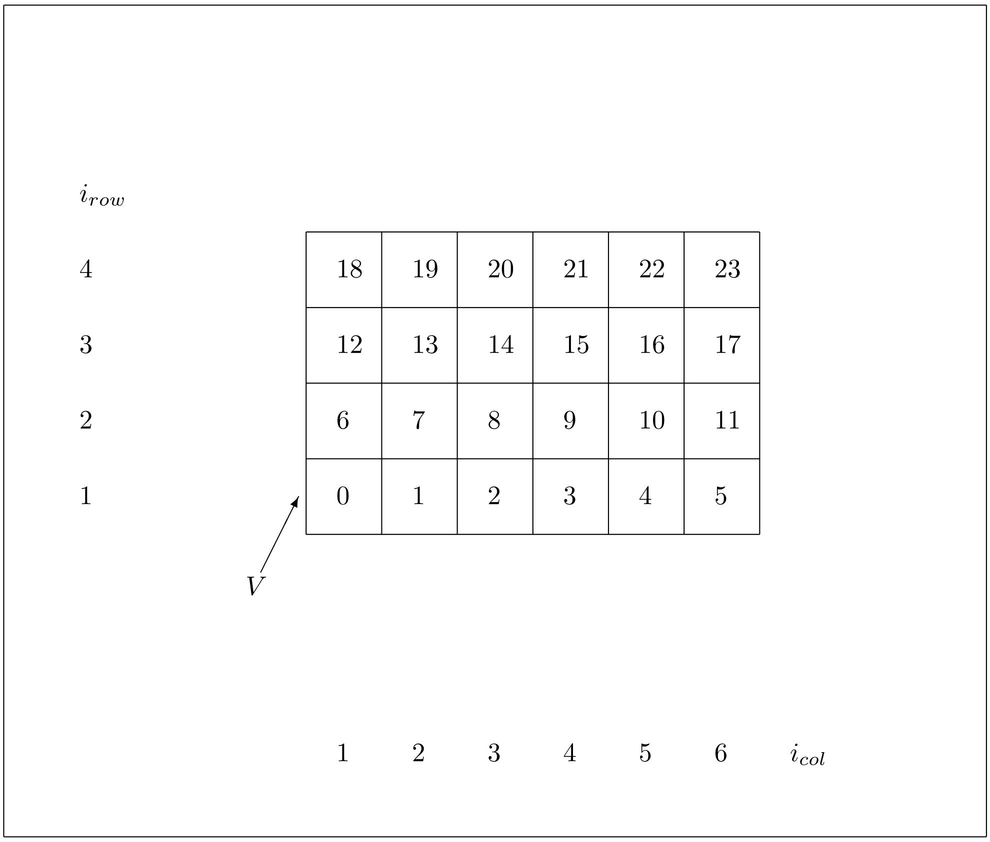


Figure 6: ANSI C - mesh XY type B

- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h_x == 1.0 [mm]`
- `double h_y == 2.0 [mm]`

11 Example of C-type mesh in ANSI C

Example of C- type mesh in ANSI C program. The mesh is analogous to A - type mesh. Just mesh mesh step $h_x = h_y = h$.

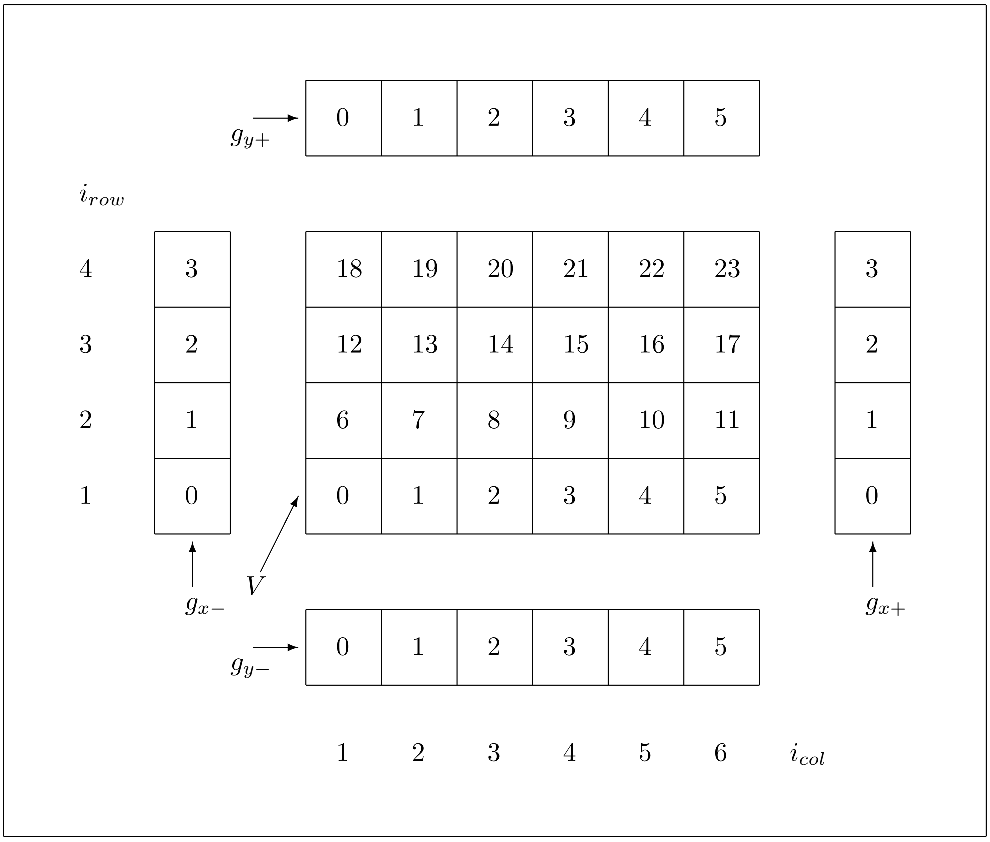


Figure 7: ANSI C - mesh XY type C

- $g_{x-} \equiv$ `double* ptr_gX_minus`
- $g_{x+} \equiv$ `double* ptr_gX_plus`
- $g_{y-} \equiv$ `double* ptr_gY_minus`
- $g_{y+} \equiv$ `double* ptr_gY_plus`
- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`

- unsigned int i_col == 1,2, .., 6
- double h == 1.0 [mm]

12 Example of D-type mesh in ANSI C

Example of D- type mesh in ANSI C program. The mesh is analogous to B - type mesh. Just $h_x = h_y = h$.

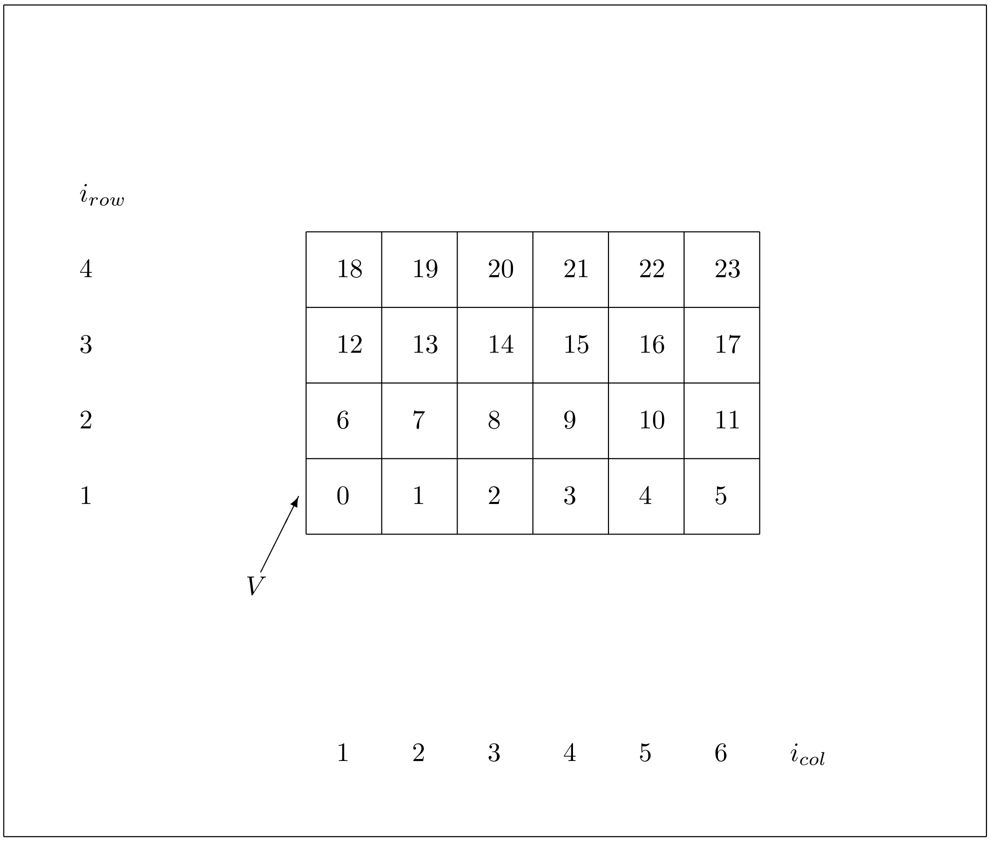


Figure 8: ANSI C - mesh XY type D

- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h == 1.0 [mm]`

13 Relaxation formula for node P1

13.1 Node description

Left, botton corner of mesh XY.

13.2 Calculation of relaxation formula

Laplace equation at node P_1

$$\nabla^2 \left(V_{(x,y)}\right)_{P_1} = 0 \tag{13.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_1} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_1} = 0 \tag{13.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_1

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_x} - \frac{V_1 - V_{1y-}}{h_x}}{h_x} = \frac{V_2 - V_1}{{h_x}^2} - \frac{g_{1x-}}{h_x} \tag{13.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_y} - \frac{V_1 - V_{1y-}}{h_y}}{h_y} = \frac{V_4 - V_1}{{h_y}^2} - \frac{g_{1y-}}{h_y} \tag{13.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_2 - V_1}{{h_x}^2} - \frac{g_{1x-}}{h_x} + \frac{V_4 - V_1}{{h_y}^2} - \frac{g_{1y-}}{h_y} = 0 \tag{13.5}$$

Let us find V_1

$$V_1 = ? \tag{13.6}$$

$$\frac{V_2 - V_1}{h_x^2} + \frac{V_4 - V_1}{h_y^2} = \frac{g_{1x-}}{h_x} + \frac{g_{1y-}}{h_y} \tag{13.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{13.8}$$

We obtain

$$V_2 h_y^2 - V_1 h_y^2 + V_4 h_x^2 - V_1 h_x^2 = g_{1x-} h_x h_y^2 + g_{1y-} h_x^2 h^y \tag{13.9}$$

$$V_1 (h_x^2 + h_y^2) = V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y \tag{13.10}$$

13.3 Final forms of relaxation formula

13.3.1 xyLV_RELAX5_P1_A

$$
\begin{aligned}
h_x &\neq h_y \\
g_{1x-}, g_{1y-} &\neq 0 \\
V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y}{h_x^2 + h_y^2}
\end{aligned}
\tag{13.11}
$$

13.3.2 xyLV_RELAX5_P1_B

$$
\begin{aligned}
h_x &\neq h_y \\
g_{1x-}, g_{1y-} &= 0 \\
V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2}{h_x^2 + h_y^2}
\end{aligned}
\tag{13.12}
$$

13.3.3 xyLV_RELAX5_P1_C

$$
\begin{aligned}
h_x = h_y &= h \\
g_{1x-}, g_{1y-} &\neq 0 \\
V_1 &= \frac{V_2 + V_4 - g_{1x-} h - g_{1y-} h}{2}
\end{aligned}
\tag{13.13}
$$

13.3.4 xyLV_RELAX5_P1_D

$$
\begin{aligned}
h_x = h_y &= h \\
g_{1x-}, g_{1y-} &= 0 \\
V_1 &= \frac{V_2 + V_4}{2}
\end{aligned}
\tag{13.14}
$$

14 Relaxation formula for node P2

14.1 Node description

Bottom edge of mesh XY.

14.2 Calculation of relaxation formula

Laplace equation at node P_2

$$\nabla^2 \left(V_{(x,y)}\right)_{P_2} = 0 \tag{14.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_2} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_2} = 0 \tag{14.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_2

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_2} \approx \frac{\frac{V_3-V_2}{h_x} - \frac{V_2-V_1}{h_x}}{h_x} = \frac{V_1+V_3-2V_2}{h_x^2} \tag{14.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_2} \approx \frac{\frac{V_5-V_2}{h_y} - \frac{V_2-V_{2y-}}{h_y}}{h_y} = \frac{V_5-V_2}{h_y^2} - \frac{g_{2y-}}{h_y} \tag{14.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_1+V_3-2V_2}{{h_x}^2} + \frac{V_5-V_2}{{h_y}^2} - \frac{g_{2y-}}{h_y} = 0 \tag{14.5}$$

Let us find V_2

$$V_2 = ? \tag{14.6}$$

$$\frac{V_1+V_3-2V_2}{h_x^2} + \frac{V_5-V_2}{h_y^2} = \frac{g_{2y-}}{h_y} \tag{14.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{14.8}$$

We obtain

$$V_1 h_y^2 + V_3 h_y^2 - 2V_2 h_y^2 + V_5 h_x^2 = g_{2y-} h_x^2 h_y \tag{14.9}$$

$$V_2\left(h_x^2 + h_y^2\right) = \left(V_1+V_3\right) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y \tag{14.10}$$

14.3 Final forms of relaxation formula

14.3.1 xyLV_RELAX5_P2_A

$$
\begin{aligned}
h_x &\neq h_y \\
g_{2y-} &\neq 0 \\
V_2 &= \frac{(V_1 + V_3)\, h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y}{h_x^2 + h_y^2}
\end{aligned}
\tag{14.11}
$$

14.3.2 xyLV_RELAX5_P2_B

$$
\begin{aligned}
h_x &\neq h_y \\
g_{2y-} &= 0 \\
V_2 &= \frac{(V_1 + V_3)\, h_y^2 + V_5 h_x^2}{h_x^2 + h_y^2}
\end{aligned}
\tag{14.12}
$$

14.3.3 xyLV_RELAX5_P2_C

$$
\begin{aligned}
h_x = h_y &= h \\
g_{2y-} &\neq 0 \\
V_2 &= \frac{V_1 + V_3 + V_5 - g_{2y-} h}{3}
\end{aligned}
\tag{14.13}
$$

14.3.4 xyLV_RELAX5_P2_D

$$
\begin{aligned}
h_x = h_y &= h \\
g_{2y-} &= 0 \\
V_2 &= \frac{V_1 + V_3 + V_5}{3}
\end{aligned}
\tag{14.14}
$$

15 Relaxation formula for node P3

15.1 Node description

Right, botton corner of mesh XY.

15.2 Calculation of relaxation formula

Laplace equation at node P_3

$$\nabla^2 \left(V_{(x,y)}\right)_{P_3} = 0 \tag{15.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_3} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_3} = 0 \tag{15.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_3

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_3} \approx \frac{\frac{V_{3x+} - V_3}{h_x} - \frac{V_3 - V_2}{h_x}}{h_x} = \frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} \tag{15.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_y} - \frac{V_3 - V_{3y-}}{h_y}}{h_y} = \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} \tag{15.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} = 0 \tag{15.5}$$

Let us find V_3

$$V_3 =? \tag{15.6}$$

$$\frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} = \frac{g_{3y-}}{h_y} - \frac{g_{3x+}}{h_x} \tag{15.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{15.8}$$

We obtain

$$V_2 h_y^2 - V_3 h_y^2 + V_6 h_x^2 - V_3 h_x^2 = g_{3y-} h_x^2 h_y - g_{3x+} h_x h_y^2 \tag{15.9}$$

$$V_3 \left(h_x^2 + h_y^2\right) = V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y \tag{15.10}$$

15.3 Final forms of relaxation formula

15.3.1 xyLV_RELAX5_P3_A

$$h_x \neq h_y$$
$$g_{3x+}, g_{3y-} \neq 0$$
$$V_3 = \frac{V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y}{h_x^2 + h_y^2} \tag{15.11}$$

15.3.2 xyLV_RELAX5_P3_B

$$h_x \neq h_y$$
$$g_{3x+}, g_{3y-} = 0$$
$$V_3 = \frac{V_2 h_y^2 + V_6 h_x^2}{h_x^2 + h_y^2} \tag{15.12}$$

15.3.3 xyLV_RELAX5_P3_C

$$h_x = h_y = h$$
$$g_{3x+}, g_{3y-} \neq 0$$
$$V_3 = \frac{V_2 + V_6 + g_{3x+} h - g_{3y-} h}{2} \tag{15.13}$$

15.3.4 xyLV_RELAX5_P3_D

$$h_x = h_y = h$$
$$g_{3x+}, g_{3y-} = 0$$
$$V_3 = \frac{V_2 + V_6}{2} \tag{15.14}$$

16 Relaxation formula for node P4

16.1 Node description

Left edge of mesh XY.

16.2 Calculation of relaxation formula

Laplace equation at node P_4

$$\nabla^2 \left(V_{(x,y)}\right)_{P_4} = 0 \tag{16.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_4} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_4} = 0 \tag{16.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_4

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_4} \approx \frac{\frac{V_5 - V_4}{h_x} - \frac{V_4 - V_{4x-}}{h_x}}{h_x} = \frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} \tag{16.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_y} - \frac{V_4 - V_1}{h_y}}{h_y} = \frac{V_1 + V_7 - 2V_4}{h_y^2} \tag{16.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = 0 \tag{16.5}$$

Let us find V_4

$$V_4 = ? \tag{16.6}$$

$$\frac{V_5 - V_4}{h_x^2} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = \frac{g_{4x-}}{h_x} \tag{16.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{16.8}$$

We obtain

$$V_5 h_y^2 - V_4 h_y^2 + V_1 h_x^2 + V_7 h_x^2 - 2V_4 h_x^2 = g_{4x-} h_x h_y^2 \tag{16.9}$$

$$V_4 \left(2h_x^2 + h_y^2\right) = \left(V_1 + V_7\right) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2 \tag{16.10}$$

16.3 Final forms of relaxation formula

16.3.1 xyLV_RELAX5_P4_A

$$h_x \neq h_y$$
$$g_{4x-} \neq 0$$
$$V_4 = \frac{\left(V_1 + V_7\right) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2}{2h_x^2 + h_y^2} \tag{16.11}$$

16.3.2 xyLV_RELAX5_P4_B

$$h_x \neq h_y$$
$$g_{4x-} = 0$$
$$V_2 = \frac{\left(V_1 + V_7\right) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \tag{16.12}$$

16.3.3 xyLV_RELAX5_P4_C

$$h_x = h_y = h$$
$$g_{4x-} \neq 0$$
$$V_4 = \frac{V_1 + V_5 + V_7 - g_{4x-} h}{3} \tag{16.13}$$

16.3.4 xyLV_RELAX5_P4_D

$$h_x = h_y = h$$
$$g_{4x-} = 0$$
$$V_4 = \frac{V_1 + V_5 + V_7}{3} \tag{16.14}$$

17 Relaxation formula for node P5

17.1 Node description

Node inside a mesh XY.

17.2 Calculation of relaxation formula

Laplace equation at node P_5

$$\nabla^2 \left(V_{(x,y)}\right)_{P_5} = 0 \tag{17.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_5} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_5} = 0 \tag{17.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_5

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_x} - \frac{V_5 - V_4}{h_x}}{h_x} = \frac{V_4 + V_6 - 2V_5}{h_x^2} \tag{17.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_y} - \frac{V_5 - V_2}{h_y}}{h_y} = \frac{V_2 + V_8 - 2V_5}{h_y^2} \tag{17.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \tag{17.5}$$

Let us find V_5

$$V_5 = ? \tag{17.6}$$

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \tag{17.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{17.8}$$

We obtain

$$V_4 h_y^2 + V_6 h_y^2 - 2V_5 h_y^2 + V_2 h_x^2 + V_8 h_x^2 - 2V_5 h_x^2 = 0 \tag{17.9}$$

$$2V_5 \left(h_x^2 + h_y^2\right) = \left(V_2 + V_8\right) h_x^2 + \left(V_4 + V_6\right) h_y^2 \tag{17.10}$$

17.3 Final forms of relaxation formula

17.3.1 xyLV_RELAX5_P5_A

$$h_x \neq h_y$$

No gradients g inside mesh are considered.

$$V_5 = \frac{(V_2 + V_8)\,h_x^2 + (V_4 + V_6)\,h_y^2}{2\left(h_x^2 + h_y^2\right)} \tag{17.11}$$

17.3.2 xyLV_RELAX5_P5_B

$$h_x \neq h_y$$

Relaxation formula is the same as xyLV_RELAX5_P5_A

$$V_5 = \frac{(V_2 + V_8)\,h_x^2 + (V_4 + V_6)\,h_y^2}{2\left(h_x^2 + h_y^2\right)} \tag{17.12}$$

17.3.3 xyLV_RELAX5_P5_C

$$h_x = h_y = h$$

No gradients g inside mesh are considered.
The formula simplifies, so no g and h terms are necessary.

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \tag{17.13}$$

17.3.4 xyLV_RELAX5_P5_D

$$h_x = h_y = h$$

The formula also simplifies.

Relaxation formula is the same as xyLV_RELAX5_P5_C

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \tag{17.14}$$

18 Relaxation formula for node P6

18.1 Node description

Right edge of mesh XY.

18.2 Calculation of relaxation formula

Laplace equation at node P_6

$$\nabla^2 \left(V_{(x,y)}\right)_{P_6} = 0 \tag{18.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_6} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_6} = 0 \tag{18.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_6

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_6} \approx \frac{\frac{V_{6x+}-V_6}{h_x} - \frac{V_6-V_5}{h_x}}{h_x} = \frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} \tag{18.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_6} \approx \frac{\frac{V_9-V_6}{h_y} - \frac{V_6-V_3}{h_y}}{h_y} = \frac{V_3 + V_9 - 2V_6}{h_y^2} \tag{18.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = 0 \tag{18.5}$$

Let us find V_6

$$V_6 = ? \tag{18.6}$$

$$\frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = -\frac{g_{6x+}}{h_x} \tag{18.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{18.8}$$

We obtain

$$V_5 h_y^2 - V_6 h_y^2 + V_3 h_x^2 + V_9 h_x^2 - 2V_6 h_x^2 = -g_{6x+} h_x h_y^2 \tag{18.9}$$

$$V_6 \left(2h_x^2 + h_y^2\right) = \left(V_3 + V_9\right) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2 \tag{18.10}$$

18.3 Final forms of relaxation formula

18.3.1 xyLV_RELAX5_P6_A

$$h_x \neq h_y$$
$$g_{6x+} \neq 0$$
$$V_6 = \frac{(V_3 + V_9)\, h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2}{2h_x^2 + h_y^2} \tag{18.11}$$

18.3.2 xyLV_RELAX5_P6_B

$$h_x \neq h_y$$
$$g_{6x+} = 0$$
$$V_6 = \frac{(V_3 + V_9)\, h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \tag{18.12}$$

18.3.3 xyLV_RELAX5_P6_C

$$h_x = h_y = h$$
$$g_{6x+} \neq 0$$
$$V_6 = \frac{V_3 + V_5 + V_9 + g_{6x+} h}{3} \tag{18.13}$$

18.3.4 xyLV_RELAX5_P6_D

$$h_x = h_y = h$$
$$g_{6x+} = 0$$
$$V_6 = \frac{V_3 + V_5 + V_9}{3} \tag{18.14}$$

19 Relaxation formula for node P7

19.1 Node description

Left, upper corner of mesh XY.

19.2 Calculation of relaxation formula

Laplace equation at node P_7

$$\nabla^2 \left(V_{(x,y)} \right)_{P_7} = 0 \tag{19.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} = 0 \tag{19.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_7

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_x} - \frac{V_7 - V_{7x-}}{h_x}}{h_x} = \frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} \tag{19.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} \approx \frac{\frac{V_{7y+} - V_7}{h_y} - \frac{V_7 - V_4}{h_y}}{h_y} = \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} \tag{19.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} + \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} = 0 \tag{19.5}$$

Let us find V_7

$$V_7 = ? \tag{19.6}$$

$$\frac{V_8 - V_7}{h_x^2} + \frac{V_4 - V_7}{h_y^2} = \frac{g_{7x-}}{h_x} - \frac{g_{7y+}}{h_y} \tag{19.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{19.8}$$

We obtain

$$V_8 h_y^2 - V_7 h_y^2 + V_4 h_x^2 - V_7 h_x^2 = g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \tag{19.9}$$

$$V_7 \left(h_x^2 + h_y^2 \right) = V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \tag{19.10}$$

19.3 Final forms of relaxation formula

19.3.1 xyLV_RELAX5_P7_A

$$
\begin{aligned}
h_x &\neq h_y \\
g_{7x-}, g_{7y+} &\neq 0 \\
V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 + g_{7y+} h_x^2 h_y}{\left(h_x^2 + h_y^2\right)}
\end{aligned}
\tag{19.11}
$$

19.3.2 xyLV_RELAX5_P7_B

$$
\begin{aligned}
h_x &\neq h_y \\
g_{7x-}, g_{7y+} &= 0 \\
V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2}
\end{aligned}
\tag{19.12}
$$

19.3.3 xyLV_RELAX5_P7_C

$$
\begin{aligned}
h_x = h_y &= h \\
g_{7x-}, g_{7y+} &\neq 0 \\
V_7 &= \frac{V_4 + V_8 - g_{7x-} h + g_{7y+} h}{2}
\end{aligned}
\tag{19.13}
$$

19.3.4 xyLV_RELAX5_P7_D

$$
\begin{aligned}
h_x = h_y &= h \\
g_{7x-}, g_{7y+} &= 0 \\
V_7 &= \frac{V_4 + V_8}{2}
\end{aligned}
\tag{19.14}
$$

20 Relaxation formula for node P8

20.1 Node description

Upper edge of mesh XY.

20.2 Calculation of relaxation formula

Laplace equation at node P_8

$$\nabla^2 \left(V_{(x,y)}\right)_{P_8} = 0 \tag{20.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_8} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_8} = 0 \tag{20.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_8

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_x} - \frac{V_8 - V_7}{h_x}}{h_x} = \frac{V_7 + V_9 - 2V_8}{h_x^2} \tag{20.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_8} \approx \frac{\frac{V_{8y+} - V_8}{h_y} - \frac{V_8 - V_5}{h_y}}{h_y} = \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} \tag{20.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} = 0 \tag{20.5}$$

Let us find V_8

$$V_8 =? \tag{20.6}$$

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} = -\frac{g_{8y+}}{h_y} \tag{20.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{20.8}$$

We obtain

$$V_7 h_y^2 + V_9 h_y^2 - 2V_8 h_y^2 + V_5 h_x^2 - V_8 h_x^2 = -g_{8y+} h_x^2 h_y \tag{20.9}$$

$$V_8 \left(h_x^2 + 2h_y^2\right) = \left(V_7 + V_9\right) h_y^2 + V_5 h_x^2 + g_{8y+} h_x^2 h_y \tag{20.10}$$

20.3 Final forms of relaxation formula

20.3.1 xyLV_RELAX5_P8_A

$$
\begin{aligned}
h_x &\neq h_y \\
g_{8y+} &\neq 0 \\
V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9)\, h_y^2 + g_{8y+} h_x^2 h_y}{h_x^2 + 2h_y^2}
\end{aligned} \tag{20.11}
$$

20.3.2 xyLV_RELAX5_P8_B

$$
\begin{aligned}
h_x &\neq h_y \\
g_{8y+} &= 0 \\
V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9)\, h_y^2}{h_x^2 + 2h_y^2}
\end{aligned} \tag{20.12}
$$

20.3.3 xyLV_RELAX5_P8_C

$$
\begin{aligned}
h_x &= h_y = h \\
g_{8y+} &\neq 0 \\
V_8 &= \frac{V_5 + V_7 + V_9 + g_{8y+} h}{3}
\end{aligned} \tag{20.13}
$$

20.3.4 xyLV_RELAX5_P8_D

$$
\begin{aligned}
h_x &= h_y = h \\
g_{8y+} &= 0 \\
V_8 &= \frac{V_5 + V_7 + V_9}{3}
\end{aligned} \tag{20.14}
$$

21 Relaxation formula for node P9

21.1 Node description

Right, upper corner of mesh XY.

21.2 Calculation of relaxation formula

Laplace equation at node P_9

$$\nabla^2 \left(V_{(x,y)}\right)_{P_9} = 0 \tag{21.1}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_9} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_9} = 0 \tag{21.2}$$

Approximation of partial derivatives of $V_{(x,y)}$ at node P_9

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2}\right)_{P_9} \approx \frac{\frac{V_{9x+} - V_9}{h_x} - \frac{V_9 - V_8}{h_x}}{h_x} = \frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} \tag{21.3}$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2}\right)_{P_9} \approx \frac{\frac{V_{9y+} - V_9}{h_y} - \frac{V_9 - V_6}{h_y}}{h_y} = \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} \tag{21.4}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} + \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} = 0 \tag{21.5}$$

Let us find V_9

$$V_9 = ? \tag{21.6}$$

$$\frac{V_8 - V_9}{h_x^2} + \frac{V_8 - V_9}{h_x^2} = -\frac{g_{9x+}}{h_x} - \frac{g_{9y+}}{h_y} \tag{21.7}$$

Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \tag{21.8}$$

We obtain

$$V_8 h_y^2 - V_9 h_y^2 + V_6 h_x^2 - V_9 h_x^2 = -g_{9x+} h_x h_y^2 - g_{9y+} h_x^2 h_y \tag{21.9}$$

$$V_9 \left(h_x^2 + h_y^2\right) = V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y \tag{21.10}$$

21.3 Final forms of relaxation formula

21.3.1 xyLV_RELAX5_P9_A

$$h_x \neq h_y$$
$$g_{9x+}, g_{9y+} \neq 0$$
$$V_9 = \frac{V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y}{h_x^2 + h_y^2} \tag{21.11}$$

21.3.2 xyLV_RELAX5_P9_B

$$h_x \neq h_y$$
$$g_{9x+}, g_{9y+} = 0$$
$$V_9 = \frac{V_6 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \tag{21.12}$$

21.3.3 xyLV_RELAX5_P9_C

$$h_x = h_y = h$$
$$g_{9x+}, g_{9y+} \neq 0$$
$$V_9 = \frac{V_6 + V_8 + g_{9x+} h + g_{9y+} h}{2} \tag{21.13}$$

21.3.4 xyLV_RELAX5_P9_D

$$h_x = h_y = h$$
$$g_{9x+}, g_{9y+} = 0$$
$$V_9 = \frac{V_6 + V_8}{2} \tag{21.14}$$

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